Collisional damping of the collective oscillations of a trapped Fermi gas

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We consider a Fermi gas confined by a harmonic trapping potential and we highlight the role of the Fermi–Dirac statistics by studying frequency and damping of collective oscillations of quadrupole type in the framework of the quantum Boltzmann equation, in which statistical corrections are taken into account in the collisional integral. We are able to describe the crossover from the collisionless regime to the hydrodynamic one by introducing a temperature-dependent relaxation time τ_Q. We show that, in the degenerate regime, the relaxation rate τ_Q^{-1} exhibits a temperature dependence different from the collision rate γ. We finally compare the collisional properties of the Fermi gas with the ones of the Bose gas for temperatures above the Bose–Einstein condensation.

I. INTRODUCTION

Since the experimental realization of Bose–Einstein condensation in trapped metastable atomic gases [1] the investigation of collective oscillations has proven to be an important tool to characterize the behaviour of these many body systems. Indeed, the high accuracy of frequency measurements [2] and the sensitivity of collective phenomena to interaction effects (see [3] and references therein for a recent discussion of collective excitations in condensed Bose gases at finite temperature) makes them good candidates to check the predictions of theory [4]. The quantum degenerate regime has also been recently reached in a magnetically trapped Fermi gas [5] and measurements of collective oscillations might be available soon also for fermionic species.

The aim of this paper is to discuss the temperature dependence of the frequency and damping of the low energy collective oscillations of a trapped Fermi gas in its normal, non superfluid phase. The transition temperature to the superfluid phase is predicted to occur at very low temperature [6,7], while the effects of quantum degeneracy should be visible before the transition [8–11]. For a discussion of the oscillations in the superfluid phase see [7,12]. Collective oscillations can occur in the collisionless ($\omega\tau_Q \gg 1$) or in the hydrodynamic ($\omega\tau_Q \ll 1$) regime. Here ω is the frequency of the oscillation and τ_Q is a characteristic relaxation time. We calculate the relaxation time τ_Q by linearizing the Boltzmann equation and we show that it is important to use τ_Q instead of the inverse of the collision rate γ, which measures the time between collisions. The two times are proportional in the classical regime, but exhibit drastically different T dependences in conditions of quantum degeneracy. In addition to the study of the temperature dependence of τ_Q in the Fermi case, which depicts the role of quantum statistics, we also compare the relaxation time of the collective oscillations with the one exhibited by a Bose gas above the critical temperature T_c for Bose–Einstein condensation, to highlight the role of the bosonic statistics.

The Pauli exclusion principle dramatically changes the collisional properties of Fermi gases in the quantum degenerate regime and the onset of degeneracy can be traced down by looking at the corresponding changes of collective oscillations. At very low temperature Fermi–Dirac statistics quenches the number of collisions between atoms and one expects to find the gas in the collisionless (CL) regime. At higher temperatures there is the possibility of finding the gas in the hydrodynamic (HD) regime, especially if the cross section is very large, as in the case of ^{6}Li. An example of such a crossover has been given in [13] for the spin dipole oscillation.

The Fermi gas is taken to be a two component ("spin up" and "spin down") gas, to avoid the absence of s-wave collisions in the sample. The Bose gas will be instead considered to be trapped in a single spin state. We use a method recently developed by Guéry-Odelin *et al.* in [14] to evaluate in relaxation time approximation the collisional integral. The relaxation time τ_Q is obtained through a linearization of the Boltzmann equation evaluated with an ansatz which describes the deformations in momentum space driven by the external oscillation. A similar technique has also been developed by Al Khawaja *et al.* [15] for the evaluation of the collective excitations of a non-condensed Bose gas. An analytical and numerical investigation of the relaxation time and of the collision rate in a non-condensed uniform Bose gas was done by Lopez-Arias and Smerzi in [16] while Kavoulakis *et al.* [17] presented a microscopic calculation of the relaxation time of a trapped non-condensed Bose gas based on the Boltzmann equation. Our approaches lead, in fact, to the same expression for the relaxation time.

The paper is organized as follows: in Section II we briefly describe the method of the averages for a two component Fermi gas and we introduce the relaxation time τ_Q for the quadrupolar collective oscillation and compare it with

the collisional rate γ. Section III is dedicated to the comparison with bosons. Section IV reports the results for the collective oscillations of the Fermi gas as a function of the temperature. Section V draws conclusions and perspectives.

II. RELAXATION TIME AND COLLISION RATE

The equations describing the dynamical evolution of the quadrupole oscillations in a two component Fermi system can be derived starting from the Boltzmann equation

$$\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\boldsymbol{\nabla}+\mathbf{F}_\sigma\cdot\boldsymbol{\nabla}_{\mathbf{p}_1}\right)f_\sigma(\mathbf{r},\mathbf{p}_1,t)=I_{coll}[f_\sigma] \tag{1}$$

for the phase space distribution function for a given spin component σ, $f_\sigma(\mathbf{r},\mathbf{p},t)$ [18] where $\mathbf{F}_\sigma=-\boldsymbol{\nabla}V^\sigma_{ext}(\mathbf{r})$ is the force originating from the external trapping potential, that we choose to be axially symmetric and that is, in general, different for different spin components

$$V^\sigma_{ext}(\mathbf{r})=\frac{1}{2}m\left[\omega^2_{\sigma,\perp}\left(x^2+y^2\right)+\omega^2_{\sigma,z}z^2\right]. \tag{2}$$

The collisional integral reads

$$I_{coll}[f_\sigma]=\frac{\sigma_f}{4\pi h^3 m}\int d^3p_2\,d\Omega\,|\mathbf{p}_1-\mathbf{p}_2|\times \\ \left[(1-f_\sigma(\mathbf{p}_1))(1-f_{-\sigma}(\mathbf{p}_2))f_\sigma(\mathbf{p}'_1)f_{-\sigma}(\mathbf{p}'_2)-f_\sigma(\mathbf{p}_1)f_{-\sigma}(\mathbf{p}_2)(1-f_\sigma(\mathbf{p}'_1))(1-f_{-\sigma}(\mathbf{p}'_2))\right]. \tag{3}$$

We have used the notation $f_\sigma(\mathbf{p}_i)\equiv f_\sigma(\mathbf{r},\mathbf{p}_i,t)$. The moments $\mathbf{p}'_1$ and $\mathbf{p}'_2$ are constrained by energy and momentum conservation: $p_1^2+p_2^2=p_1'^2+p_2'^2$ and $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}'_1+\mathbf{p}'_2$. As the gas is at very low temperature the collisional cross section is taken to be $\sigma_f=4\pi a^2$ with a the s-wave scattering length for up-down collisions. These collisions are the dominant ones at low temperatures, as up-up and down-down collisions occur in the p channel for antisymmetry requirements [19]. By neglecting p-wave scattering, suppressed by the centrifugal barrier at the low energies of interest [20], we allow interactions only between up and down particles.

The Boltzmann equation (1) can be used to describe the trapped gas if one can use the semiclassical approximation and if mean field contributions to the energy of the gas are negligible. The former condition reads $k_BT\gg\hbar\omega_{\perp,z}$ and can be fulfilled over a wide range of temperatures for large number of trapped particles as the Fermi temperature, below which quantum degeneracy effects begin to show, is defined as $T_F=(3N)^{1/3}\hbar\omega_{ho}/k_B$ (we have introduced the geometrical average of the trapping frequencies $\omega_{ho}=(\omega_\perp^2\omega_z)^{1/3}$). The condition for the semiclassical approximation is then $T/T_F\gg(3N)^{-1/3}$ and is ensured if N is sufficiently large. Mean field effects are small the case for fermions [13] and this is also true for the Bose gas above the critical temperature for Bose–Einstein condensation $T_c=[N/\zeta(3)]^{1/3}\hbar\omega_{ho}/k_B$. The reason is that the gas is dilute and the mean field interactions, of the order of $gn(0)$ (here $g=4\pi\hbar^2a/(2m)$ with $n(0)$ the central density), must be compared to the kinetic energy which, above T_c, is of the order of k_BT. The correction is typically of the order of few percent.

When working with a two component system it is convenient to use the total distribution function $f=f_\uparrow+f_\downarrow$ and the difference $f_-=f_\uparrow-f_\downarrow$. By summing and subtracting the Boltzmann equation for the up and the down component one gets

$$\left(\partial_t+\frac{\mathbf{p}_1}{m}\cdot\boldsymbol{\nabla}+\mathbf{F}_+\cdot\boldsymbol{\nabla}_{\mathbf{p}_1}\right)f(\mathbf{r},\mathbf{p}_1,t)+\mathbf{F}_-\cdot\boldsymbol{\nabla}_{\mathbf{p}_1}f_-(\mathbf{r},\mathbf{p}_1,t)=I_{coll}[f_\uparrow]+I_{coll}[f_\downarrow] \tag{4}$$

and

$$\left(\partial_t+\frac{\mathbf{p}_1}{m}\cdot\boldsymbol{\nabla}+\mathbf{F}_+\cdot\boldsymbol{\nabla}_{\mathbf{p}_1}\right)f_-(\mathbf{r},\mathbf{p}_1,t)+\mathbf{F}_-\cdot\boldsymbol{\nabla}_{\mathbf{p}_1}f(\mathbf{r},\mathbf{p}_1,t)=I_{coll}[f_\uparrow]-I_{coll}[f_\downarrow] \tag{5}$$

where $\mathbf{F}_\pm=(\mathbf{F}_\uparrow\pm\mathbf{F}_\downarrow)/2$. In the right hand side of equations (4)-(5) the collisional contributions should be expressed in terms of f and f_-. In the simplified case of equal trapping potentials $V^\uparrow_{ext}=V^\downarrow_{ext}=V_{ext}$ for the two spin components one has $\mathbf{F}_-=0$. If also $N_\uparrow=N_\downarrow$ then the difference f_- is zero, at equilibrium, for symmetry. In this work we will restrict our attention only to in-phase oscillations. One is then left with one equation, in which the only unknown is the total phase space distribution function $f=2f_\uparrow=2f_\downarrow$.

A systematic way to solve the Boltzmann equation is provided by the Chapman–Enskog expansion [21] which allows to introduce the viscosity and the thermal conductivity in the hydrodynamic equations that will then exhibit

damping. To actually calculate the viscosity and the thermal conductivity it is necessary to provide some ansatz for their functional dependence.

Starting from equation (4) with $f_- = 0$ we instead derive equations for the average of generic dynamical quantities $\chi(\mathbf{r}, \mathbf{p})$:

$$\partial_t \langle \chi \rangle - m^{-1} \langle \mathbf{p} \cdot \boldsymbol{\nabla} \chi \rangle - \langle \mathbf{F} \cdot \boldsymbol{\nabla}_{\mathbf{p}} \chi \rangle = \sum_\sigma \langle \chi I_{coll}[f_\sigma] \rangle \tag{6}$$

where the average is taken both in position and momentum space

$$\langle \chi \rangle = \int \chi(\mathbf{r}, \mathbf{p}) f(\mathbf{r}, \mathbf{p}) \, d\Gamma \Big/ \int f(\mathbf{r}, \mathbf{p}) \, d\Gamma \tag{7}$$

having defined $d\Gamma = d^3r \, d^3p/h^3$. The collisional contribution can be rewritten in the form

$$\sum_\sigma \langle \chi I_{coll}[f_\sigma] \rangle = \frac{1}{4} \sum_\sigma \int \Delta\chi(\mathbf{r}, \mathbf{p}) I_{coll} \left[f_\sigma(\mathbf{r}, \mathbf{p}) \right] d\Gamma \Big/ \int f(\mathbf{r}, \mathbf{p}) \, d\Gamma \tag{8}$$

where $\Delta\chi = \chi_1 + \chi_2 - \chi_{1'} - \chi_{2'}$ with $\chi_i \equiv \chi(\mathbf{r}, \mathbf{p}_i)$. If χ is a quantity conserved during the elastic collision then integral (8) is zero. This happens if $\chi = a(\mathbf{r}) + \mathbf{b}(\mathbf{r}) \cdot \mathbf{p} + c(\mathbf{r}) p^2$ with a, $\mathbf{b}$ and c arbitrary functions of the position.

As shown in [14] the dynamical evolution of the $m_z = 0$ quadrupole oscillation in an axially symmetric trapping potential is described by a system of six coupled equations that we rewrite here:

$$\begin{aligned}
&\partial_t \langle \chi_1 \rangle - 2 \langle \chi_3 \rangle = 0 \\
&\partial_t \langle \chi_2 \rangle - 2 \langle \chi_4 \rangle = 0 \\
&\partial_t \langle \chi_3 \rangle - \langle \chi_5 \rangle + \frac{2\omega_\perp^2 + \omega_z^2}{3} \langle \chi_1 \rangle + \frac{\omega_z^2 - \omega_\perp^2}{3} \langle \chi_2 \rangle = 0 \\
&\partial_t \langle \chi_4 \rangle - \langle \chi_6 \rangle + \frac{2\omega_z^2 - 2\omega_\perp^2}{3} \langle \chi_1 \rangle + \frac{\omega_\perp^2 + 2\omega_z^2}{3} \langle \chi_2 \rangle = 0 \\
&\partial_t \langle \chi_5 \rangle + \frac{2\omega_z^2 + 4\omega_\perp^2}{3} \langle \chi_3 \rangle + \frac{2\omega_z^2 - 2\omega_\perp^2}{3} \langle \chi_4 \rangle = 0 \\
&\partial_t \langle \chi_6 \rangle + \frac{4\omega_z^2 - 4\omega_\perp^2}{3} \langle \chi_3 \rangle + \frac{4\omega_z^2 + 2\omega_\perp^2}{3} \langle \chi_4 \rangle = \sum_\sigma \langle \chi_6 I_{coll}[f_\sigma] \rangle .
\end{aligned} \tag{9}$$

The χ_i quantities are defined as

$$\begin{aligned}
&\chi_1 = r^2 && \chi_2 = 2z^2 - r_\perp^2 \\
&\chi_3 = \mathbf{r} \cdot \mathbf{v} && \chi_4 = 2 z v_z - \mathbf{r}_\perp \cdot \mathbf{v}_\perp \\
&\chi_5 = v^2 && \chi_6 = 2 v_z^2 - v_\perp^2
\end{aligned} \tag{10}$$

where $\mathbf{v} = \mathbf{p}/m$ and $\mathbf{r}_\perp$, $\mathbf{v}_\perp$ are respectively the projections of the $\mathbf{r}$ and $\mathbf{v}$ vectors in the xy plane. The $m_z = 0$ quadrupole oscillation is coupled to the monopole one unless the trap is spherical ($\omega_z = \omega_\perp$). The χ_6 combination is the only one for which the collisional contribution does not vanishes. To describe the $m_z = 2$ quadrupole oscillation one should instead introduce the mean values of xy, $v_x y + v_y x$ and $v_x v_y$. We shall not consider here this mode.

Solving system (9) is still a hard task because a full solution of the Boltzmann equation is needed to actually calculate the average of the collisional contribution. What can be done is to linearize the distribution function assuming a specific form for its deviation from equilibrium, finding a variational estimate for $f(\mathbf{r}, \mathbf{p})$. By recalling that in the collisionless regime one can have solutions with anisotropic equilibrium velocities, corresponding to different temperatures in the radial and axial directions, we make the ansatz

$$f_\sigma(\mathbf{r}, \mathbf{p}) = f_0(\mathbf{r}, (1 + \alpha_\perp)^{1/2} \mathbf{p}_\perp, (1 + \alpha_z)^{1/2} p_z) \tag{11}$$

where f_0 is the equilibrium distribution function. The proposed anisotropy in momentum space is of quadrupole type if one chooses $\alpha_\perp = -\alpha_z/2$. In this case the normalization is preserved to first order in the deformation parameters. Deformations in the real space do not need to be taken into account in ansatz (11) because, after linearization, they do not contribute to the integral as a consequence of the invariance property of the collisional integral already discussed.

The linearization of f_σ around $\alpha_\perp = \alpha_z = 0$ leads to

$$f_\sigma \simeq f_0 - \left(\frac{\alpha_\perp \beta p_\perp^2}{2m} + \frac{\alpha_z \beta p_z^2}{2m} \right) f_0(1-f_0) \tag{12}$$

with $\beta = (k_B T)^{-1}$ and, using the equilibrium identity $(1-f_0(\mathbf{p}_1))(1-f_0(\mathbf{p}_2))f_0(\mathbf{p}_1')f_0(\mathbf{p}_2') = f_0(\mathbf{p}_1)f_0(\mathbf{p}_2)(1-f_0(\mathbf{p}_1'))(1-f_0(\mathbf{p}_2'))$, we find the following expression, to first order in the alpha's, for the collisional term:

$$\sum_\sigma \langle \chi_6 I_{coll}[f_\sigma] \rangle \simeq \frac{\sigma_f \beta}{16\pi h^6 m N} \int d^3r d^3p_1 d^3p_2 d\Omega \Delta\chi_6 |\mathbf{v}_1 - \mathbf{v}_2| \times$$
$$\left(\alpha_\perp \Delta p_\perp^2 + \alpha_z \Delta p_z^2\right) f_0(\mathbf{p}_1) f_0(\mathbf{p}_2)(1-f_0(\mathbf{p}_1'))(1-f_0(\mathbf{p}_2')). \tag{13}$$

Carrying out the calculation (see Appendix) shows that the collisional contribution is proportional to the mean value of $\langle \chi_6 \rangle$ so that one can write

$$\sum_\sigma \langle \chi_6 I_{coll}[f_\sigma] \rangle = -\frac{\langle \chi_6 \rangle}{\tau_Q} \tag{14}$$

defining the temperature dependent relaxation time for the quadrupole oscillation. The dimensionless quantity $\omega_{ho}\tau_Q$ can be written in the form

$$\frac{1}{\omega_{ho}\tau_Q} = \frac{4}{5\sqrt[3]{3}\pi} \left(N^{1/3} \frac{a}{a_{ho}} \right)^2 F_Q\left(\frac{T}{T_F}\right) \tag{15}$$

where $a_{ho} = [\hbar/(m\omega_{ho})]^{1/2}$ is the harmonic oscillator length and $F_Q(t)$ (see Appendix) is an universal function of the reduced temperature $t = T/T_F$, showed in Figure 1, which contains the effect of Fermi statistics. Numerical coefficients have been chosen so that F_Q has the limiting form $1/t$ for $t \gg 1$ (classical regime). In the fully degenerate regime the integral defining F_Q is analytical and the result

$$F_Q(t) = 8\pi^2 t^2 \tag{16}$$

is found. The T^{-2} behaviour for the relaxation time is typical of degenerate Fermi systems [22,23]. We recall that result (16) holds above the BCS phase transition. Conversely we point out explicitly that result (15) in the classical limit is correct only as long as the approximations that led to the collisional integral (3) remain valid. The truly classical system is the one for which all partial waves are taken into account in the two body collision, thereby restoring the collisions between up-up and down-down particles, no longer suppressed by (quantum) antisymmetry requirements. The classical limit of equation (15) instead keeps on having the quantal characteristic of up-down s-wave scattering only. However p-wave collisions in real experiments become important only at temperatures much higher than the Fermi temperature [20] and it is safe to describe in such a way our system. We finally observe that the relaxation time depends on the square of the scattering length a and, consequently, it is not sensitive to its sign. Mean field corrections, that have been neglected in the Boltzmann equation, do depend on the sign of the interaction [13].

Using result (14) the system of equations (9) is closed and the dispersion law for the collective oscillations can be obtained. Before considering the oscillation frequencies we address our attention to the number of collisions that take place in the gas by calculating the collision rate

$$\gamma = \frac{2\sigma_f}{4\pi N h^6} \int d^3r d^3p_1 d^3p_2 d\Omega \, |\mathbf{v}_1 - \mathbf{v}_2| f_{1\uparrow} f_{2\downarrow} (1-f_{1'\uparrow})(1-f_{2'\downarrow}). \tag{17}$$

The factor 2 in equation (17) takes into account the presence of the two components. By treating integral (17) in the very same way of the collisional contribution (15) we find the following expression for the rate at equilibrium (put in adimensional form):

$$\frac{\gamma}{\omega_{ho}} \equiv \frac{1}{\omega_{ho}\tau_\gamma} = \frac{1}{\sqrt[3]{3}\pi} \left(N^{1/3} \frac{a}{a_{ho}} \right)^2 F_\gamma\left(\frac{T}{T_F}\right) \tag{18}$$

where again the coefficients of the function F_γ are chosen so that it has the $1/t$ classical behaviour. In the degenerate regime one instead finds the analytical result

$$\frac{1}{\omega_{ho}\tau_\gamma} \underset{T\to 0}{\to} 3^{2/3} 8\pi \left(N^{1/3} \frac{a}{a_{ho}} \right)^2 \left(\frac{T}{T_F}\right)^3. \tag{19}$$

Result (19) explicitly shows the importance of using the relaxation time τ_Q instead of the inverse of the collision rate τ_γ: while in the classical regime they are proportional, their ratio being $(\tau_Q/\tau_\gamma)_{class.} = 5/4$, in the degenerate regime their temperature dependence is different. The cubic dependence on temperature of the collision rate in the quantum degenerate regime is not a consequence of the trapping but holds also in the uniform system where the same type of analytical calculations can be carried on as well.

III. COMPARISON WITH THE BOSE CASE

The absence of proportionality between collision rate and relaxation time in a non condensed Bose gas was pointed out by Lopez-Arias and Smerzi [16] who numerically calculated the relaxation time τ_Q for the uniform gas. The method of averages allows to calculate τ_Q for the trapped Bose gas by considering the same ansatz (11) used in the Fermi case. We now have only one spin component and thus the spin index is absent. The cross section is $\sigma_b = 8\pi a^2$ because the scattering is between indistinguishable particles and the collisional integral reads

$$I_{coll}[f] = \frac{\sigma_b}{4\pi h^3 m}\int d^3p_2\, d\Omega\, |\mathbf{p}_1 - \mathbf{p}_2| \times \\ [(1+f(\mathbf{p}_1))(1+f(\mathbf{p}_2))f(\mathbf{p}_1')f(\mathbf{p}_2') - f(\mathbf{p}_1)f(\mathbf{p}_2)(1+f(\mathbf{p}_1'))(1+f(\mathbf{p}_2'))]\,. \quad (20)$$

The $(1+f)$ factors in the above equation account for the increase of probability of a scattering event in an already occupied final state, typical of Bose systems.

The expression for the quadrupolar relaxation time τ_Q^b, analog to (15) and in agreement with [17], is given by

$$\frac{1}{\omega_{ho}\tau_Q^b} = \frac{16\sqrt[3]{\zeta(3)}}{5\pi}\left(N^{1/3}\frac{a}{a_{ho}}\right)^2 G_Q\left(\frac{T}{T_c}\right) \quad (21)$$

and the one for the collision rate, analog to (18), reads instead

$$\frac{\gamma_b}{\omega_{ho}} \equiv \frac{1}{\omega_{ho}\tau_{\gamma_b}} = \frac{4\sqrt[3]{\zeta(3)}}{\pi}\left(N^{1/3}\frac{a}{a_{ho}}\right)^2 G_{\gamma_b}\left(\frac{T}{T_c}\right). \quad (22)$$

The functions G_Q and G_{γ_b} are plotted in Figure 2 and their expressions are given in Appendix. Both functions behave as T_c/T in the classical limit. Comparison of the two equations shows again the limiting ratio $(\tau_Q^b/\tau_{\gamma_b})_{class.} = 5/4$ [14]. By using an improved trial function the authors of [17] have shown that in the classical limit the ratio changes by only 1%, proving the validity of ansatz (11).

The quantum behaviour of τ_Q^b and τ_{γ_b} is opposite to the behaviour of the corresponding fermionic quantities. Quantum statistical effects enhance the number of collisions taking place in the trapped gas and the rate γ_b is seen to increase with respect to the classical trend. The integral G_{γ_b} of equation (22) (see Appendix) is well behaved as $\beta\mu \to 0^-$ and reaches a finite value, differently from the uniform case [16], where the collision rate diverges at the critical temperature, driving to zero the time between collisions. This divergence has also been found by Nikuni *et al.* [24], in the framework of a quantum Boltzmann equation coupled with an equation for the condensate, as T_c is approached both from above and from below. In the non-uniform case instead τ_{γ_b} does not vanishes at the transition. To understand this it is useful to recall that τ_{γ_b}, as well as τ_Q^b and their fermionic counterparts, can be regarded as the average over the whole sample of a local collision (relaxation) time. It is then natural to find a weakening of the divergences exhibited by uniform systems. Figure 2 shows that the quadrupolar relaxation rate $(\tau_Q^b)^{-1}$, similarly to γ_b, increases its value with respect to the classical value but in a less stronger way as compared to the increase of the collision rate. This witnesses the loss of efficiency of the collisions near the transition temperature: the number of collisions gets larger, due to the Bose factors $(1+f)$, but the relaxation rate does not increases its value proportionally [25]. This feature was pointed out also in [16] where the authors even find a diverging behaviour for the relaxation time of the quadrupolar oscillation. To obtain a diverging behaviour in our trapped case one should observe the vanishing of the function G_Q in equation (21) as $T \to T_c^+$. This is not the case, as the integral is positive defined and increases its value near the critical temperature. As final remark we observe that the (linearized) Boltzmann equation is not accurate in describing the system at the Bose–Einstein phase transition because it completely neglects fluctuations and coherence phenomena so that one should not look at the quantum limit $\beta\mu = 0$ as a quantitative prediction. Conversely the behaviour of G_Q and G_{γ_b} as $T \to T_c^+$ is physically relevant. The coupled dynamics of the condensate and the thermal component in the trapped case below T_c can be found in [26] where the normal component and the Bose condensed fraction are taken into account in the framework of a quantum Boltzmann equation coupled

with a generalized Gross–Pitaevskii equation. Finally two examples of theories that also deal with the issue of the condensate formation, within formalism derived from quantum optics and from field theory, can be found in [27] and [28].

IV. QUADRUPOLE OSCILLATIONS IN THE FERMI GAS

We now consider the frequency and the damping of the quadrupolar collective oscillation that can be derived from the set of equations (9) together with (14). By looking for solution of oscillatory type one finds the dispersion relation by imposing the determinant of the system to be zero. The dispersion relation coincides with the one already derived for the classical gas [14], as quantum statistical effects are kept into account in the relaxation time τ_Q. One gets

$$\left(\omega^2 - 4\omega_z^2\right)\left(\omega^2 - 4\omega_\perp^2\right) - \frac{i}{\omega\tau_Q}\left(\omega^4 - \frac{2}{3}\omega^2\left(5\omega_\perp^2 + 4\omega_z^2\right) + 8\omega_\perp^2\omega_z^2\right) = 0. \tag{23}$$

The first term of (23) corresponds to the dispersion law for the pure collisionless regime ($\omega\tau_Q \to \infty$). In this case the eigenfrequencies coincide with the ones predicted by the single particle harmonic oscillator Hamiltonian: $\omega_{CL} = 2\,\omega_{\perp,z}$. The total Hamiltonian is in fact the sum of single particle Hamiltonians and interactions are completely neglected. In this case the concept of collective oscillation actually looses his meaning. On the other hand the term multiplying $i/(\omega\tau_Q)$ refers to the pure hydrodynamical regime ($\omega\tau_Q \to 0$) and the dispersion relation gives the result [29] $\omega_{HD}^2/\omega_\perp^2 = (4\lambda^2 + 5 \pm \sqrt{16\lambda^4 - 32\lambda^2 + 25})/3$ where $\lambda = \omega_z/\omega_\perp$ is the anisotropy parameter. Putting $\lambda = 1$ gives two frequencies $\omega_{HD}^M = 2\,\omega_{ho}$ and $\omega_{HD}^Q = \sqrt{2}\,\omega_{ho}$ corresponding to the monopole and quadrupole mode respectively (as observed, system (9) decouples in two sub-sets). The relevant limits for deformed traps $\lambda \ll 1$ and $\lambda \gg 1$ can be obtained as well [14,15]. When the relaxation time τ_Q is of the order of the oscillation frequency one is in an intermediate regime and equation (23) is solved numerically for $\omega = \omega_r + i\Gamma$ yielding the frequency ω_r and the damping Γ of the oscillation. As observed in [30] in the Bose gas case, the notion of hydrodynamic regime in trapped gases fails at the boundary of the trap, where the gas is rarefied and collisions are rare. The average performed over all the sample in calculating the relaxation time τ_Q takes care of this effect. Indeed the possibility of an unified description of the collisionless and of the hydrodynamic regimes is the useful feature of the presented model.

By looking at the temperature dependence of the relaxation rate, depicted in Figure 1, it is clear that the gas will be in the collisionless regime both at low and high temperatures. At low temperatures the Pauli principle reduces the number of available final states for the scattering process and the relaxation rate tends to zero as $(T/T_F)^2$. At high temperatures, as noted previously, the relaxation rate is proportional to the collision rate and decreases as T_F/T. This is basically a density effect: for fixed number of particles a Boltzmann gas gets more and more rarefied as temperature increases. In the intermediate region there is room for an increase of τ_Q^{-1}. Whether the relaxation time τ_Q will reach a sufficiently small value to bring the system in the hydrodynamic regime depends on the combination $(N^{1/3}a/a_{ho})^2$ of the number of particles, the scattering length and the trap parameters of equation (15) as the function F_Q depends on the reduced temperature only. The maximum of F_Q is ~ 1.8 at $T/T_F \simeq 0.35$ so that a minimum value of $\omega_{ho}\tau_Q = 3.15/(N^{1/3}a/a_{ho})^2$ is reached. Large numbers of trapped atoms as well as large scattering lengths should allow the observation of the hydrodynamic regime. In the following we will consider explicitly the relevant experimental cases of ^{40}K [5] and ^{6}Li [31].

The crossover from collisionless to the hydrodynamic regime is most striking in the case of the spin dipole oscillation [13] where the dispersion law is simpler than (23) and allows to discriminate analytically between a damped oscillatory behaviour and an overdamped one. In the case of spin dipole oscillations one can find overdamping because the spin current $\mathbf{v}_\uparrow - \mathbf{v}_\downarrow$ is not conserved during collisions and this leads to a purely diffusive mode in the collision dominated regime. In the case of quadrupole oscillations instead both the collisionless and the hydrodynamic regimes support an oscillation and the actual regime can be deduced from the value of the real part of the oscillation frequency which passes, *e.g.* in the spherical case, from $2\,\omega_{ho}$ to $\sqrt{2}\,\omega_{ho}$. The spectrum of a trapped Fermi gas in the hydrodynamic regime has been studied independently by Amoruso *et al.* [32] and by Bruun and Clark [33] for $T \ll T_F$ and $T \gg T_F$. The frequency of the quadrupolar oscillation clearly coincides with result (23) in the $\omega\tau_Q \to 0$ limit. However we observe that in the limits of low and high temperature the value of the oscillation frequency will not be the hydrodynamic but the collisionless one. This even supplies a possible signature for the superfluid transition since the value of the quadrupole frequency for a superfluid Fermi gas coincides with the hydrodynamic value [12] and at the calculated temperature for the BCS phase transition [6,7] τ_Q is large and the system is in the collisionless regime. So for large number of particles and for large values of $|a|/a_{ho}$ one should observe, as a function of the decreasing temperature, the sequence of collisionless – hydrodynamic – collisionless – hydrodynamic value for the real part of the quadrupole frequency and a corresponding structure of local maxima and minima in the damping rate of the oscillation.

To summarize this discussion we show in Figure 3 the real and the imaginary part of the quadrupolar oscillation of a spherical ($\lambda = 1$) cloud of ^{40}K atoms. The scattering length is [5] $a = 157\, a_0$ with a_0 the Bohr radius. We take the ratio $a/a_{ho} \sim 10^{-3}$ and $N = 10^6$. The minimum value of $\omega_{ho}\tau_Q$ is then ~ 350 and the system is always in the collisionless regime. In the upper panel the real part of the frequency (solid line) is compared with the frequency of a Boltzmann gas (dashed line) which, at low temperatures, is always in the hydrodynamic regime. This emphasizes the role of Fermi statistics in the low temperature regime of the trapped gas. The lower panel shows the damping of the oscillation. The two gases exhibit zero damping in the pure collisionless regime at high temperature. At zero temperature both the Boltzmann gas and the Fermi gas have zero damping but the former is in the pure hydrodynamic regime while the latter is in the pure collisionless one because of Pauli blocking. Inspection of the damping rate gives then another clear signature of quantum degeneracy as it departs significantly from the classical value.

Figure 4 shows instead the real (upper panel) and the imaginary (lower panel) part of the quadrupolar oscillation frequency for $N = 10^6$ atoms of ^{6}Li, that have a resonant value of the scattering length $a = -2160\, a_0$, in a spherical trap. Due to the lighter mass with respect to potassium the ratio $|a|/a_{ho}$ is $\sim 5 \times 10^{-2}$ and a minimum value of $\omega_{ho}\tau_Q \sim 0.14$ is found. Lithium can thus be found in the hydrodynamic regime in the region around the maximum of the function F_Q (Figure 1). Indeed the real part of the frequency bends towards the hydrodynamic value $\sqrt{2}\,\omega_{ho}$ as long as the Fermi statistics allows a sufficient collisional activity. Correspondingly the damping has a relative minimum. One should point out that if the system had reached the pure hydrodynamical regime at $T \simeq 0.35\, T_F$ the damping would have then been exactly zero. As the temperature is further lowered Pauli principle forces the system to become again collisionless, the real part raises again towards the value $2\,\omega_{ho}$ while the damping exhibit a local maximum and then falls to zero once the complete collisionless regime is reached.

The two proposed examples involve the measured scattering lengths for potassium and lithium. Different predictions for the frequency and the damping can be obtained for different values of the number of trapped atoms, as well as of the trapping parameters and scattering length through the use of the definition (15) of the relaxation time together with the universal function plotted in Figure 1. A way to reach the hydrodynamic regime also in the case of the potassium isotope could be the change of the value of its scattering length with external fields, as already experimentally observed in alkali atoms [34].

V. CONCLUSIONS

We have studied the collective frequencies and the damping of a trapped normal Fermi system through the use of a linearized Boltzmann equation, a method that gives a variational estimate for the relaxation time τ_Q relative to the coupled monopole-quadrupole oscillations. The effects of Fermi statistics are included in the relaxation time τ_Q that diverges at zero temperature as a consequence of the Pauli principle and at classical temperatures $T > T_F$ as a consequence of the diluitness of the gas. In both cases the collisionless regime for the collective oscillation is reached. At intermediate temperatures we have found the possibility of a hydrodynamical behaviour of the quadrupole mode for the ^{6}Li isotope, due to his large scattering length, for reasonable number of trapped particles. We have also compared the relaxation rate τ_Q^{-1} with the collision rate γ, finding analytically different behaviours at low temperatures. This emphasizes the need of distinguishing τ_Q from γ^{-1}. The role of quantum statistics has also been considered in the case of a Bose gas and the difference between the collision rate and the relaxation rate have been pointed out also in this case. Future developments will concentrate on non symmetric configurations, that may be relevant from the experimental point of view, as well as on the study of the damping of collective oscillations in mixed systems.

Useful discussions with A. Smerzi and H. T. C. Stoof are acknowledged. I wish to thank S. Stringari for critically reading the manuscript.

APPENDIX A: CALCULATION OF THE COLLISIONAL INTEGRAL

We first introduce the center of mass variables

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{V} + \frac{\mathbf{v}}{2} \qquad & \mathbf{v}_1' &= \mathbf{V} + \frac{\mathbf{v}'}{2} \\ \mathbf{v}_2 &= \mathbf{V} - \frac{\mathbf{v}}{2} \qquad & \mathbf{v}_2' &= \mathbf{V} - \frac{\mathbf{v}'}{2} \end{aligned} \tag{A1}$$

where $|\mathbf{v}| = |\mathbf{v}'| = v$ for momentum conservation. The Jacobian associated to transformation (A1) is one. In these variables one has

$$\begin{aligned} \Delta v_\perp^2 &= -\frac{1}{2}\left(v_z^2 - v_z'^2\right) \\ \Delta v_z^2 &= \frac{1}{2}\left(v_z^2 - v_z'^2\right) \end{aligned} \tag{A2}$$

and $\Delta\chi_6 = 2\Delta v_z^2 - \Delta v_\perp^2$. The product $f_0(\mathbf{v}_1)f_0(\mathbf{v}_2)(1-f_0(\mathbf{v}_1'))(1-f_0(\mathbf{v}_2'))$ can be written in the useful form

$$f_0(\mathbf{v}_1)f_0(\mathbf{v}_2)(1-f_0(\mathbf{v}_1'))(1-f_0(\mathbf{v}_2')) = \frac{1}{4}\frac{1}{\cosh(\nu)+\cosh(\gamma\mathbf{V}\cdot\mathbf{v})}\frac{1}{\cosh(\nu)+\cosh(\gamma\mathbf{V}\cdot\mathbf{v}')} \tag{A3}$$

where $\nu = \beta(m(V^2+v^2/4)/2 + m\omega_\perp^2(x^2+y^2+\lambda^2z^2)/2 - \mu)$ and $\gamma = \beta m/2$. We then have to linear order in $\alpha_\perp$, α_z

$$\sum_\sigma \langle \chi_6 I_{coll}[f_\sigma]\rangle = \frac{3\sigma_f \beta m^7 \delta\alpha}{2^8\pi h^6 N} \times I \tag{A4}$$

with $\delta\alpha = \alpha_z - \alpha_\perp$ and

$$I = \int d^3r d^3V d^3v d\Omega\, v\left(v_z^2 - v_z'^2\right)^2 \frac{1}{\cosh(\nu)+\cosh(\gamma\mathbf{V}\cdot\mathbf{v})}\cdot\frac{1}{\cosh(\nu)+\cosh(\gamma\mathbf{V}\cdot\mathbf{v}')}. \tag{A5}$$

Integral (A5) can be carried on as follows: introduce a reference frame $(\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}})$ oriented such that $\mathbf{V} = V\hat{\mathbf{a}}$ and $\hat{\mathbf{z}}$ is in the plane generated by $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. We then have the following expressions:

$$v_z = v_a(\hat{\mathbf{a}}\cdot\hat{\mathbf{z}}) + v_b(\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}) = v\cos(\vartheta)(\hat{\mathbf{a}}\cdot\hat{\mathbf{z}}) + v\sin(\vartheta)\cos(\varphi)(\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}) \tag{A6}$$

$$v_z' = v_a'(\hat{\mathbf{a}}\cdot\hat{\mathbf{z}}) + v_b'(\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}) = v\cos(\vartheta')(\hat{\mathbf{a}}\cdot\hat{\mathbf{z}}) + v\sin(\vartheta')\cos(\varphi')(\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}) \tag{A7}$$

and $\mathbf{V}\cdot\mathbf{v}^{(\prime)} = Vv\cos(\vartheta^{(\prime)})$. Integrations in $d\varphi$ and $d\varphi'$ are immediate. One can then integrate in $d\vartheta$ and $d\vartheta'$ using the fact that $\cosh(x)$ is an even function to end with the following intermediate form of the integral

$$I = \frac{2^7\pi^4}{5\lambda}\int r^2 dr V^2 dV v^7 dv \int_{-1}^{1} dxdy\left(\frac{1}{3}+\frac{2}{3}x^2 - x^2y^2\right)\frac{1}{\cosh(\nu)+\cosh(\gamma Vvx)}\cdot\frac{1}{\cosh(\nu)+\cosh(\gamma Vvy)}. \tag{A8}$$

A final change of variables to polar dimensionless coordinates leads to the final form of I which has to be integrated numerically except in the low and high temperature limits:

$$\begin{aligned} I = \frac{2^{21}\pi^4}{5\omega_{ho}^3}\frac{1}{(\beta m)^7}\int_0^\infty \rho^6 d\rho \int_0^{\frac{\pi}{2}} d\vartheta d\varphi \cos^2(\vartheta)\sin^{10}(\vartheta)\sin^7(\varphi)\cos^2(\varphi)\times \\ \int_{-1}^{1} d\eta d\xi\left(\frac{1}{3}+\frac{2}{3}\eta^2-\eta^2\xi^2\right)\frac{1}{\cosh(\rho-\beta\mu)+\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\eta)}\times \\ \frac{1}{\cosh(\rho-\beta\mu)+\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\xi)}. \end{aligned} \tag{A9}$$

The value of $\langle\chi_6\rangle$ can be calculated to be, to first order in $\delta\alpha$

$$\langle\chi_6\rangle = -\frac{4\delta\alpha}{Nm\beta^4(\hbar\omega_{ho})^3} f_4(z) \tag{A10}$$

and the common proportionality of (A5) and (A10) on $\delta\alpha$ allows to write equation (14) of the text. Using (A10) and (A9) one builds the F_Q function as

$$\begin{aligned} F_Q\left(\frac{T}{T_F}\right) = \frac{9\,2^5}{f_4(z)}\left(\frac{T}{T_F}\right)^2\int_0^\infty \rho^6 d\rho\int_0^{\frac{\pi}{2}} d\vartheta d\varphi\cos^2(\vartheta)\sin^{10}(\vartheta)\sin^7(\varphi)\cos^2(\varphi)\times \\ \int_{-1}^{1} d\eta d\xi\left(\frac{1}{3}+\frac{2}{3}\eta^2-\eta^2\xi^2\right)\frac{1}{\cosh(\rho-\beta\mu)+\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\eta)}\times \\ \frac{1}{\cosh(\rho-\beta\mu)+\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\xi)}. \end{aligned} \tag{A11}$$

In the classical limit $\beta\mu \to -\infty$ and the denominators such as $\cosh((\rho-\beta\mu)+\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\xi)$ can be approximated with $\exp(\rho-\beta\mu)/2$. In the quantum degenerate regime instead $\beta\mu \to \infty$ and, after changing variables to $\rho' = (\rho-\beta\mu)$, $\xi' = (\rho'+\beta\mu)\sin^2(\vartheta)\sin(2\varphi)\xi$ and $\eta' = (\rho'+\beta\mu)\sin^2(\vartheta)\sin(2\varphi)\eta$ one has integrals over the whole real axis that can be done analytically.

The same type of calculations can be done for the collision rate as well as for the corresponding bosonic expressions of τ_Q^b and τ_{γ_b}. The formula for F_γ is:

$$F_\gamma\left(\frac{T}{T_F}\right) = \frac{9\,2^7}{\pi}\left(\frac{T}{T_F}\right)^5 \int_0^\infty d\rho\, \frac{\rho^2}{\sinh^2(\rho-\beta\mu)} \int_0^{\pi/2} d\vartheta\, \sin^2(2\vartheta) \int_0^{\pi/2} d\varphi\, \cos(\varphi)\, \times$$
$$\mathrm{atanh}^2\left(\tanh\left((\rho-\beta\mu)/2\right)\tanh\left(\rho\sin^2(\vartheta)\sin(2\varphi)/2\right)\right). \quad \text{(A12)}$$

In the Bose case one finds the Bose function $g_4(z)$ in place of the Fermi function $f_4(z)$ and the expressions for the bosonic quadrupolar relaxation function G_Q and for the bosonic rate function G_{γ_b} are

$$G_Q\left(\frac{T}{T_c}\right) = \frac{48}{\zeta(3)\pi g_4(z)}\left(\frac{T}{T_c}\right)^2 \int_0^\infty \rho^6 d\rho \int_0^{\frac{\pi}{2}} d\vartheta d\varphi \cos^2(\vartheta)\sin^{10}(\vartheta)\sin^7(\varphi)\cos^2(\varphi)\, \times$$
$$\int_{-1}^1 d\eta d\xi \left(\frac{1}{3}+\frac{2}{3}\eta^2-\eta^2\xi^2\right) \frac{1}{\cosh(\rho-\beta\mu)-\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\eta)}\, \times$$
$$\frac{1}{\cosh(\rho-\beta\mu)-\cosh(\rho\sin^2(\vartheta)\sin(2\varphi)\xi)}, \quad \text{(A13)}$$

$$G_{\gamma_b}\left(\frac{T}{T_c}\right) = \frac{2^5}{\pi\zeta(3)^2}\left(\frac{T}{T_c}\right)^5 \int_0^\infty \frac{\rho^2\, d\rho}{\sinh^2(\rho-\beta\mu)} \int_0^{\pi/2} d\vartheta\, d\varphi\, \sin^2(2\vartheta)\cos(\varphi)\, \times$$
$$\mathrm{atanh}^2\left(\frac{\tanh(\rho\sin^2(\vartheta)\sin(2\varphi)/2)}{\tanh\left((\rho-\beta\mu)/2\right)}\right). \quad \text{(A14)}$$

The classical limit is performed again by taking $\beta\mu \to -\infty$. Differently from the Fermi case, as the critical temperature T_c is approached the fugacity tends to one, *i.e.* $\beta\mu \to 0$. In this limit the integrand in (A14) diverges in the origin as a logarithm but the integral is finite. This is a consequence of the confinement, since in the uniform case the collision rate diverges at the critical temperature [16,24]. Integral (A13) can be worked out analytically to include the effect of quantum degeneracy to first order in the fugacity $z = \exp(\beta\mu)$ far from the transition temperature ($t \gg 1$). We perform the expansion to check explicitly the equivalence of our expression (A13) with the one presented in [17]. By expanding the denominator in the integral in powers of the fugacity and re-introducing Cartesian coordinates we obtain

$$G_Q(t) \simeq \frac{128 t^2 z^2}{\zeta(3)\pi g_4(z)} \int_0^\infty x_3^2 dx_3 x_2^7 dx_2 x_1^2 dx_1 \int_{-1}^1 d\eta d\xi \left(1+2\eta^2-3\eta^2\xi^2\right) \times$$
$$\left\{\exp[-2(x_1^2+x_2^2+x_3^2)] + 2z\exp[-3(x_1^2+x_2^2+x_3^2)]\left(\cosh(2x_1x_2\eta)+\cosh(2x_1x_2\xi)\right)\right\} \quad \text{(A15)}$$

which gives

$$G_Q(t) \simeq \frac{t^2 z^2}{\zeta(3) g_4(z)}\left(1+\frac{3}{8}z\right). \quad \text{(A16)}$$

From the normalization condition $g_3(z) = \zeta(3)/t^3$ one obtains the correction to the fugacity

$$z \simeq \frac{\zeta(3)}{t^3} - \frac{1}{8}\left(\frac{\zeta(3)}{t^3}\right)^2 \quad \text{(A17)}$$

which, together with the expansion of $g_4(z) \simeq z + z^2/16$ lead to

$$G_Q(t) \underset{t\gg 1}{=} \frac{1}{t}\left(1+\frac{3}{16}\frac{\zeta(3)}{t^3}\right) \quad \text{(A18)}$$

which is the classical limit $1/t$ with the quantum correction of [17]. A similar expansion can be performed for the rate function G_{γ_b} as well as for the functions F_Q and F_γ. We report the result for the F_Q function which is

$$F_Q(t) \underset{t\gg 1}{=} \frac{1}{t}\left(1 - \frac{1}{32}\frac{1}{t^3}\right). \tag{A19}$$

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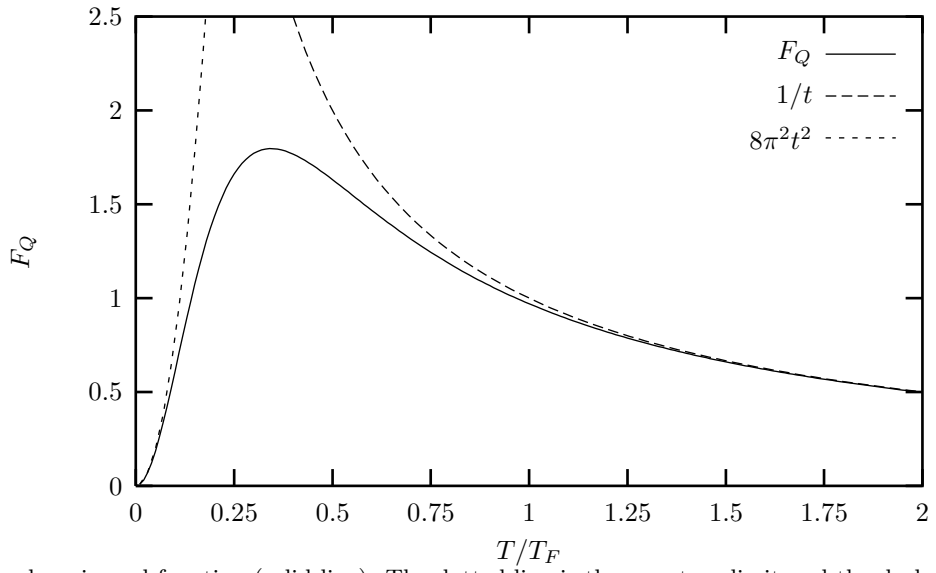

FIG. 1. Quadrupole universal function (solid line). The dotted line is the quantum limit and the dashed line is the classical prediction.

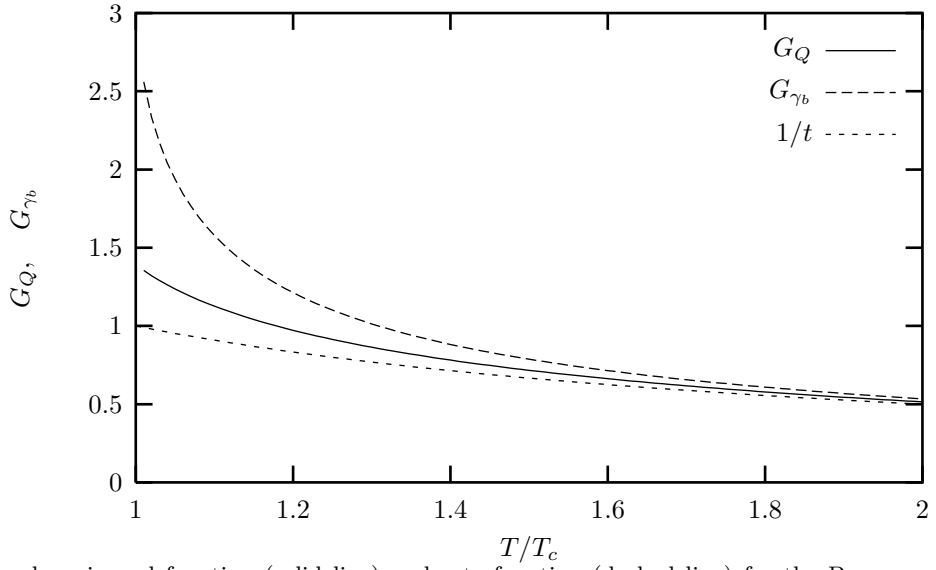

FIG. 2. Quadrupole universal function (solid line) and rate function (dashed line) for the Bose case as a function of the reduced temperature T/T_c. The classical limit $1/t$ is shown as a dotted line.

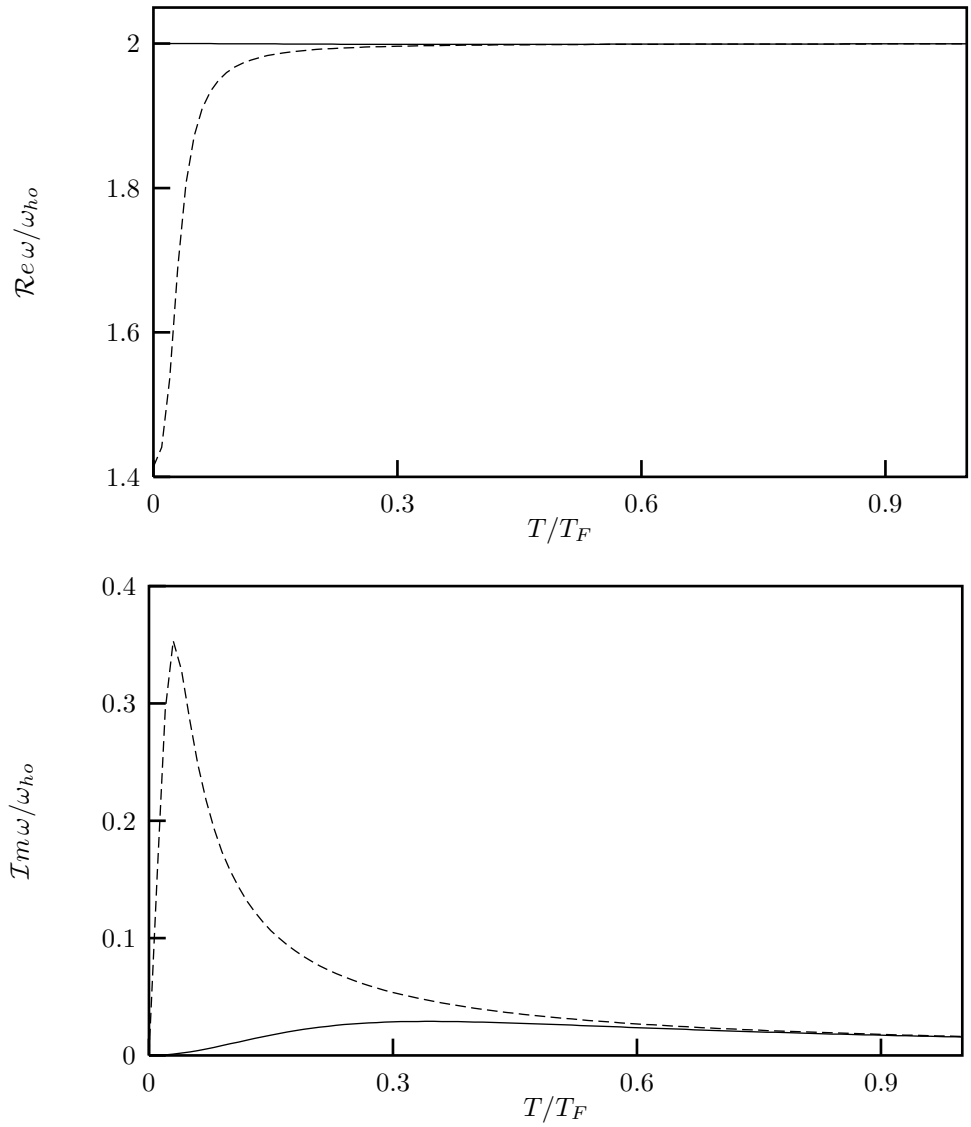

FIG. 3. Real and imaginary part of the quadrupole oscillation frequency for $N = 10^6$ ^{40}K atoms (solid lines). The dashed lines are the corresponding values for a classical gas.

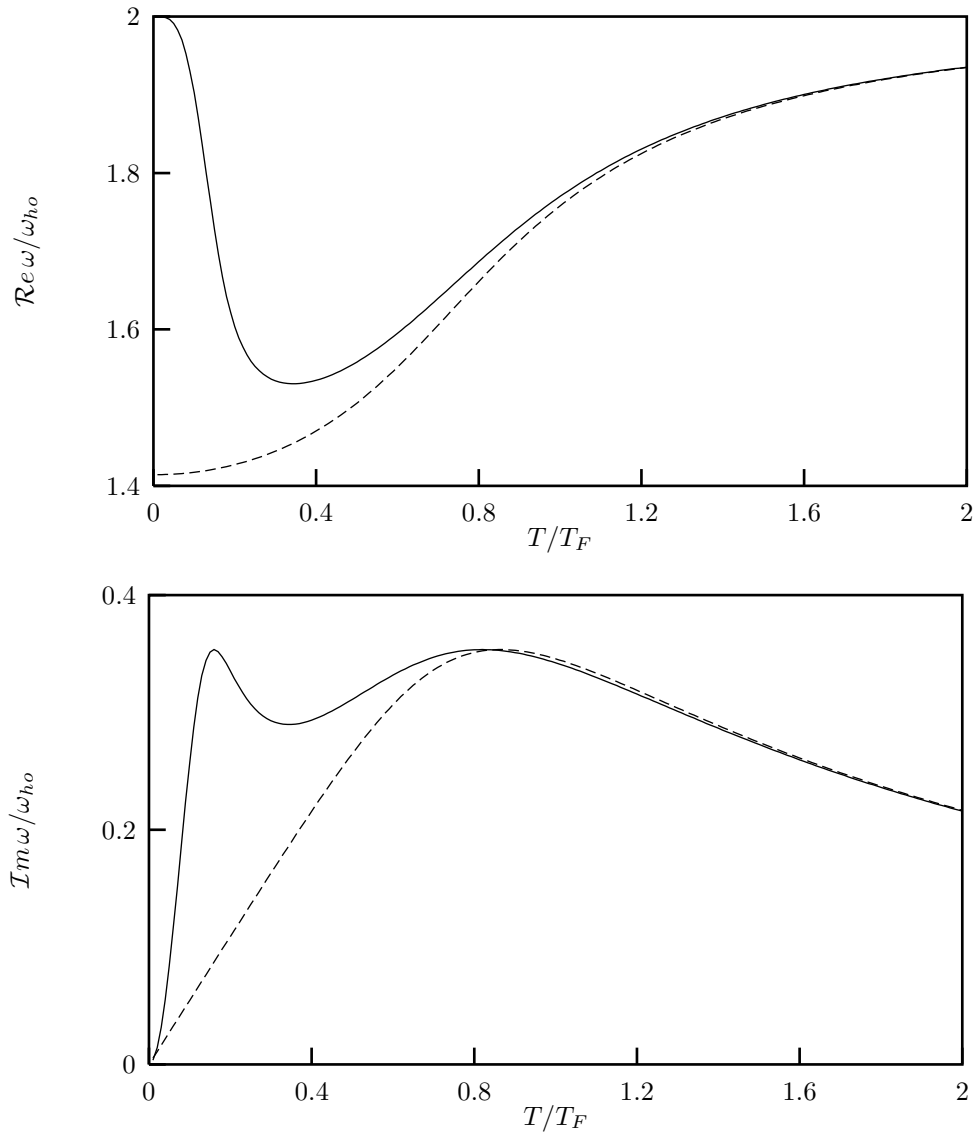

FIG. 4. Real and imaginary part of the quadrupole oscillation frequency for $N = 10^6$ ^{6}Li atoms (solid lines). The dashed lines are the corresponding values for a classical gas.